Linear Transformations

Def: Let V and W be vector spaces over the same field F. Consider a function T from V to W. T is called a linear transformation if for all vectors $\vec{v}$ and $\vec{u}$ in V and scalars α in F we have that

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

② $T(\alpha\vec{u}) = \alpha T(\vec{u})$

If $V=W$ then T is frequently called a linear operator on V.

Notation: We write $T: V \to W$ to denote that T is a function and V is the input to T and W is the output.

Ex: Consider the function $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x+2y, 3x-y, 3x)$.

- Show that T is a linear transformation by verifying axioms.
- Point out that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Ex: Consider the function $D: P_2 \to P_2$ defined by $D(a+bx+cx^2) = b + 2cx$

- Show that D is a linear transformation and point out that it is the derivative.
- Note that D can be thought of as the matrix $D\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Ex: Consider the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (x+1, y-2)$. Is T a linear transformation?

- Show that it isn't.

In fact, linear transformations on finite dimensional vector spaces and matrices are the same thing in disguise.

Def: Let V be a vector space. Suppose that $v_1, v_2, \ldots, v_n$ is a basis for V. We write $\beta = [v_1, v_2, \ldots, v_n]$ to mean that β is an <u>ordered basis</u> for V, that is order matters.

Def: Suppose that V is a vector space with ordered basis $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$. Let $\vec{x}$ be a vector in V. Write

$$\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_n \vec{v}_n$$

We write

$$[\vec{x}]_\beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

for the <u>coordinates of $\vec{x}$ with respect to β</u>.

Ex: $V = \mathbb{R}^2$, $\beta = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$.

$\vec{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$[\vec{v}]_\beta = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Modify β to $\beta' = \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$

Then $[\vec{v}]_{\beta'} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Let $\beta'' = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ be the standard basis.

Then $[\vec{v}]_{\beta''} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Ex: $V = P_2$, $\beta = [1, 1+x, 1+x+x^2]$

$\vec{x} = 2 - x + 3x^2$. $\longrightarrow [\vec{x}]_\beta = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$

$$2 - x + 3x^2 = a \cdot 1 + b(1+x) + c(1+x+x^2)$$

$$\left. \begin{aligned} 2 &= a+b+c \\ -1 &= b+c \\ 3 &= c \end{aligned} \right\} \rightarrow \begin{aligned} c &= 3 \\ b &= -4 \\ a &= 2-b-c = 2+4-3 = 3 \end{aligned}$$

Def: Let $L: V \to W$ be a linear transformation between two vector spaces V and W. Suppose that $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ is an ordered basis for V and γ is an ordered basis for W. The matrix

$$[L]_\beta^\gamma = \left([L(\vec{v}_1)]_\gamma \,\middle|\, [L(v_2)]_\gamma \,\middle|\, \cdots \,\middle|\, [L(v_n)]_\gamma \right)$$

is called the matrix for L with respect to β and γ. If $V = W$ and $\beta = \gamma$ then we write $[L]_\beta$ instead of $[L]_\beta^\gamma$.

Key fact: If everything is as above then

$$[L(\vec{v})]_\gamma = [L]_\beta^\gamma [\vec{v}]_\beta$$

for all $\vec{v}$ in V. If $\gamma = \beta$ then we have $[L(\vec{v})]_\beta = [L]_\beta [\vec{v}]_\beta$.

Ex: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$$

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$. (We will use $\gamma = \beta$).

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, $$[L]_\beta = \left(\left[L\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_\beta \middle| \left[L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_\beta \right) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

What if we change bases?

Let $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$. (again $\gamma = \beta'$)

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So, $$[L]_{\beta'} = \left(\left[L\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\beta'} \middle| \left[L\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]_{\beta'} \right) = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

What do these matrices do?

<u>Ex</u>: Take the same L, β, β' as above.

Let $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

① To use $[L]_\beta$ we need $[\vec{v}]_\beta$.

$\vec{v} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So, $[\vec{v}]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Note that

$$[L]_\beta [\vec{v}]_\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

the same

Now take a look at

$$[L(\vec{v})]_\beta = \left[L\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\beta = \left[\begin{pmatrix} 1+2 \\ 2-2 \end{pmatrix}\right]_\beta = \left[\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right]_\beta = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

② To use $[L]_{\beta'}$ we need $[\vec{v}]_{\beta'}$.

$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. So, $[\vec{v}]_{\beta'} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

Note that

$$[L]_{\beta'} [\vec{v}]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 6/4 \\ -6/4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

And

$$[L(\vec{v})]_{\beta'} = \left[\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

the same

$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$$

<u>Ex:</u> $[L]_{\beta}^{\beta'}$

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = b\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\left.\begin{matrix} a - c = 1 \\ a + c = 2 \end{matrix}\right] \rightarrow \quad 2a = 3 \rightarrow a = 3/2, \quad c = a - 1 = 1/2$$

$$\left.\begin{matrix} b - d = 1 \\ b + d = -1 \end{matrix}\right] \rightarrow \quad 2b = 0 \rightarrow \boxed{\begin{matrix} b = 0 \\ d = -1 \end{matrix}}$$

$$[L]_{\beta}^{\beta'} = \begin{pmatrix} 3/2 & 0 \\ 1/2 & -1 \end{pmatrix}$$

$$[L]_{\beta}^{\beta'}[V]_{\beta} = \begin{pmatrix} 3/2 & 0 \\ 1/2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 - 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

$$= [L(V)]_{\beta'}$$

Main idea:

$[L]_\beta$ computes $L(\vec{v})$ but it accepts the input $\vec{v}$ in terms ~~of the coordinates of $\vec{v}$ in A~~ of the β-coordinates of $\vec{v}$ and the output is $L(\vec{v})$ but in β-coordinates.

[For $[L]_\beta^\gamma$ the input is β-coordinates and the output is γ-coordinates]

We now show how to go between two coordinate systems

Def: Let V be a vector space with ordered bases β and β'. Let $I: V \to V$ be the linear transformation where $I(\vec{v}) = \vec{v}$ for all $\vec{v}$ (identity transformation). The matrix $[I]_\beta^{\beta'}$ is called the change of basis matrix from β to β'.

Let $V=\mathbb{R}^2$

Ex: Let $\beta = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$ and

$\beta' = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right]$ be as before.

Lets make the change of basis matrix from β to β'.

$$\underbrace{\begin{aligned} I\begin{pmatrix}1\\0\end{pmatrix} \\ I\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}}_{\text{feed } \beta \text{ into } I} \begin{aligned} = \\ = \end{aligned} \underbrace{\begin{aligned} \begin{pmatrix}1\\0\end{pmatrix} &= \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \tfrac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix} \\ \begin{pmatrix}0\\1\end{pmatrix} &= \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}\end{aligned}}_{\text{express the output in terms of } \beta'}$$

$$[I]_\beta^{\beta'} = \left(\left[I\begin{pmatrix}1\\0\end{pmatrix}\right]_{\beta'} \middle| \left[I\begin{pmatrix}0\\1\end{pmatrix}\right]_{\beta'} \right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Note that if $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}$ then $[\vec{v}]_\beta = \begin{pmatrix}1\\2\end{pmatrix}$

and $[I]_\beta^{\beta'}[\vec{v}]_\beta = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3/2\\1/2\end{pmatrix} = [\vec{v}]_{\beta'}$

we saw this in previous example.

Theorem: Let V be a vector space, β and β' be ordered bases for V, $L: V \to V$ be a linear transformation and $Q = [I]_{\beta}^{\beta'}$ be the change of basis matrix. Then

① Q is invertible and $Q^{-1} = [I]_{\beta'}^{\beta}$

② $[\vec{v}]_{\beta'} = Q[\vec{v}]_{\beta}$ for all $\vec{v}$ in V.

③ $[L]_{\beta} = \underbrace{Q^{-1}[L]_{\beta'}Q}_{[I]_{\beta'}^{\beta}[L]_{\beta'}[I]_{\beta}^{\beta'}}$